

SPHERICAL GEOMETRY AND INTEGRABLE SYSTEMS

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ABSTRACT. We prove that the cosine law for spherical triangles and spherical tetrahedra defines integrable systems, both in the sense of multidimensional consistency and in the sense of dynamical systems.

1. INTRODUCTION

Nowadays, it is a well accepted fact that many fundamental notions and results of differential geometry are closely related with the theory of integrable systems. Moreover, the recent development of discrete differential geometry [1] led to uncovering and identifying the true roots of such relations in the most fundamental incidence theorems of projective geometry and its metric subgeometries. On the other hand, this development led also to a clarification of the notion of integrability, which is conceptually much more clear at the level of discrete systems. In particular, the notion of multidimensional consistency as integrability crystallized in the course of this development (see [1]).

In the present paper, we offer a new evidence in favor of the thesis that the very basic structures of geometry are integrable in their nature. We show that the cosine laws for spherical simplices define integrable systems in several senses.

Our main results are as follows:

- We show that the map sending planar angles of a spherical triangle to its side lengths is integrable in the sense of multidimensional consistency (Sect. 3). Moreover, we identify this map with the symmetric reduction of the well known discrete Darboux system, which appeared for the first time in the context of discrete conjugate nets. We also discuss the relation of this result with a geometric realization of tetrahedral angle structures, due to Feng Luo [11].
- We show that the same map, interpreted as a dynamical system, is completely integrable in the Liouville-Arnold sense (Sect. 4). This map can be considered as a non-rational discretization of the famous Euler top, which is a basic example of integrable system. Moreover, we identify the second iterate of this map as a birational discretization of the Euler top, introduced by Hirota and Kimura [5] and studied in detail in our previous publication [15]. This second iterate is shown to admit a nice geometric interpretation as a particularly simple and nice map between spherical triangles, which we call *switch* (the angles of the switched triangle are set to be equal to the sides of the original one). A closely related involutive map between spherical triangles was studied in a paper by Jonas [7], which can be thus considered as an early instance of appearance of integrable systems in elementary geometry.

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- We show that the map sending dihedral angles of a spherical tetrahedron to its side lengths is a further example of an integrable dynamical system (Sect. 5). For this map, we establish conserved quantities, which turn out to be a re-interpretation of the sine laws, and an invariant volume form. The latter is shown to be closely related, via the famous Schläfli formula, to the determinant of the Hesse matrix of the volume of the spherical tetrahedron as a function of dihedral angles.

For convenience of the reader, we recall (in Sect. 2) the standard derivation of the spherical cosine and sine laws via Gram matrices.

2. COSINE LAW FOR SPHERICAL SIMPLICES

The maps we are dealing with in the present paper have their origin in the cosine law for spherical simplices. We understand a spherical $(n-1)$ -simplex as an intersection of n hemispheres in \mathbb{S}^{n-1} in general position. Let $v_1, \dots, v_n \in \mathbb{S}^{n-1}$ be the vertices of a spherical $(n-1)$ -simplex, and let $v_1^*, \dots, v_n^* \in \mathbb{S}^{n-1}$ be the vertices of the polar simplex, defined by the conditions

$$\langle v_i^*, v_j \rangle = 0 \quad \text{if } i \neq j, \quad \langle v_i^*, v_i \rangle > 0.$$

The edge lengths $\ell_{ij} \in (0, \pi)$ and the dihedral angles $\alpha_{ij} \in (0, \pi)$ are given by

$$\cos \ell_{ij} = \langle v_i, v_j \rangle, \quad \cos \alpha_{ij} = -\langle v_i^*, v_j^* \rangle.$$

Each set of data $\{\ell_{ij}\}$ or $\{\alpha_{ij}\}$ defines a spherical simplex uniquely, and the spherical cosine law gives a transition from one set of data to another. The most direct way to derive the cosine law is through the so called Gram matrices

$$G = (-\cos \alpha_{ij})_{i,j=1}^n, \quad G' = (\cos \ell_{ij})_{i,j=1}^n.$$

According to Lemma 1.2 in [10], Gram matrices of spherical simplices are characterized by the following properties: they are positive definite symmetric matrices with diagonal entries equal to 1. Introducing the matrices V and W whose columns are v_i , resp. v_i^* , we can write:

$$W^T W = G, \quad V^T V = G', \quad V^T W = D,$$

where D is a diagonal matrix with positive entries. There follows:

$$G' = D G^{-1} D, \tag{1}$$

from which the *cosine laws* follow immediately. They can be put as

$$\cos \ell_{ij} = \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}}, \quad \cos \alpha_{ij} = \frac{g'_{ij}}{\sqrt{g'_{ii}g'_{jj}}}, \tag{2}$$

where g_{ij} , g'_{ij} are the cofactors of the Gram matrices G , G' , respectively, see, for instance, [18]. We now consider in more detail the particular cases $n = 3, 4$ of this construction.

2.1. Spherical triangles. For a spherical triangle, α_{ij} is just the plane (inner) angle opposite to the edge ℓ_{ij} , see Fig. 1.

The angles of a spherical triangle belong to the tetrahedron-shaped domain

$$\mathcal{T} = \{\alpha = (\alpha_{12}, \alpha_{13}, \alpha_{23}) \in (0, \pi)^3 : \alpha_{12} + \alpha_{13} + \alpha_{23} > \pi, -\alpha_{ij} + \alpha_{ik} + \alpha_{jk} < \pi\}.$$

Here and below we use (i, j, k) to denote any permutation of $(1, 2, 3)$. The sides belong to the domain

$$\mathcal{T}^* = \{\ell = (\ell_{12}, \ell_{13}, \ell_{23}) \in (0, \pi)^3 : \ell_{12} + \ell_{13} + \ell_{23} < 2\pi, -\ell_{ij} + \ell_{ik} + \ell_{jk} > 0\},$$

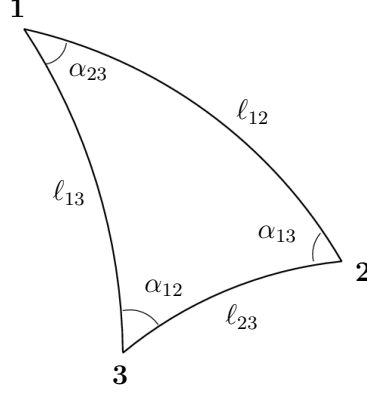


FIGURE 1. A spherical triangle.

which can be obtained from \mathcal{T} via the reflection $I : \mathcal{T} \leftrightarrow \mathcal{T}^*$, $I(\alpha) = \pi - \alpha$. The space of spherical triangles (up to orthogonal transformations of \mathbb{R}^3) is parametrized by either of \mathcal{T} or \mathcal{T}^* : a spherical triangle exists if and only if $\alpha \in \mathcal{T}$, resp. if and only if $\ell \in \mathcal{T}^*$.

The *cosine laws* for spherical triangles, as given in (2), read:

$$\cos \ell_{ij} = \frac{\cos \alpha_{ij} + \cos \alpha_{ik} \cos \alpha_{jk}}{\sin \alpha_{ik} \sin \alpha_{jk}}, \quad (3)$$

$$\cos \alpha_{ij} = \frac{\cos \ell_{ij} - \cos \ell_{ik} \cos \ell_{jk}}{\sin \ell_{ik} \sin \ell_{jk}}. \quad (4)$$

We will consider formulas (3–4) as defining the mutually inverse maps

$$\begin{aligned} \Phi : \mathcal{T} &\rightarrow \mathcal{T}^*, & \Phi^{-1} : \mathcal{T}^* &\rightarrow \mathcal{T}, \\ \alpha &\mapsto \ell, & \ell &\mapsto \alpha. \end{aligned}$$

We note that $\Phi^{-1} = I \circ \Phi \circ I$.

The cosine law can be supplemented by the *sine law* for spherical triangles:

$$\frac{\sin \ell_{12}}{\sin \alpha_{12}} = \frac{\sin \ell_{13}}{\sin \alpha_{13}} = \frac{\sin \ell_{23}}{\sin \alpha_{23}} = \frac{\sqrt{d}}{\gamma} = \frac{\gamma'}{\sqrt{d'}}, \quad (5)$$

where

$$\begin{aligned} d &= \det G = 1 - \cos^2 \alpha_{12} - \cos^2 \alpha_{13} - \cos^2 \alpha_{23} - 2 \cos \alpha_{12} \cos \alpha_{13} \cos \alpha_{23}, \\ d' &= \det G' = 1 - \cos^2 \ell_{12} - \cos^2 \ell_{13} - \cos^2 \ell_{23} + 2 \cos \ell_{12} \cos \ell_{13} \cos \ell_{23}, \end{aligned}$$

and

$$\gamma = \sin \alpha_{12} \sin \alpha_{13} \sin \alpha_{23}, \quad \gamma' = \sin \ell_{12} \sin \ell_{13} \sin \ell_{23}.$$

The sine law is derived by considering principal 2×2 minors of the relation (1).

Upon the change of variables

$$x_{ij} = \cos \alpha_{ij}, \quad y_{ij} = \cos \ell_{ij},$$

the cosine law for spherical triangles takes the form

$$y_{ij} = \frac{x_{ij} + x_{ik}x_{jk}}{\sqrt{1-x_{ik}^2}\sqrt{1-x_{jk}^2}}, \quad (6)$$

$$x_{ij} = \frac{y_{ij} - y_{ik}y_{jk}}{\sqrt{1-y_{ik}^2}\sqrt{1-y_{jk}^2}}. \quad (7)$$

We denote these maps by

$$\begin{aligned} \phi : \tau &\rightarrow \tau^*, & \phi^{-1} : \tau^* &\rightarrow \tau, \\ x &\mapsto y, & y &\mapsto x, \end{aligned}$$

and call them the algebraic forms of Φ, Φ^{-1} , respectively. They are defined on the corresponding subsets τ, τ^* of $[-1, 1]^3$, and are related by $\phi^{-1} = i \circ \phi \circ i$, $i(x) = -x$. The sine law in these new variables reads:

$$\frac{1-y_{12}^2}{1-x_{12}^2} = \frac{1-y_{13}^2}{1-x_{13}^2} = \frac{1-y_{23}^2}{1-x_{23}^2} = \frac{d}{\gamma^2} = \frac{(\gamma')^2}{d'}, \quad (8)$$

where

$$d = 1 - x_{12}^2 - x_{13}^2 - x_{23}^2 - 2x_{12}x_{13}x_{23}, \quad d' = 1 - y_{12}^2 - y_{13}^2 - y_{23}^2 + 2y_{12}y_{13}y_{23}, \quad (9)$$

and

$$\gamma^2 = (1-x_{12}^2)(1-x_{13}^2)(1-x_{23}^2), \quad (\gamma')^2 = (1-y_{12}^2)(1-y_{13}^2)(1-y_{23}^2).$$

For further reference, we mention that the sine law can be represented in either of the following forms:

$$d = (1-y_{ij}^2)(1-x_{ik}^2)(1-x_{jk}^2), \quad d' = (1-x_{ij}^2)(1-y_{ik}^2)(1-y_{jk}^2). \quad (10)$$

2.2. Spherical tetrahedra. For a spherical tetrahedron, see Fig. 2, the *cosine law* is expressed by the formula (2) with

$$g_{ij} = x_{ij} + x_{ik}x_{jk} + x_{im}x_{jm} + x_{km}(x_{ik}x_{jm} + x_{im}x_{jk} - x_{ij}x_{km}), \quad (11)$$

$$g_{ii} = 1 - (x_{jk}^2 + x_{jm}^2 + x_{km}^2) - 2x_{jk}x_{jm}x_{km}, \quad (12)$$

where we adopt the notation $x_{ij} = \cos \alpha_{ij}$ and (i, j, k, m) stands for any permutation of $(1, 2, 3, 4)$.

These formulae are supplemented (see [2, 9]) by the *sine law*, which can be put as

$$\frac{\sin \ell_{12} \sin \ell_{34}}{\sin \alpha_{12} \sin \alpha_{34}} = \frac{\sin \ell_{13} \sin \ell_{24}}{\sin \alpha_{13} \sin \alpha_{24}} = \frac{\sin \ell_{14} \sin \ell_{23}}{\sin \alpha_{14} \sin \alpha_{23}} = \frac{d}{\gamma} = \frac{\gamma'}{d'}, \quad (13)$$

and by the formulas

$$\begin{aligned} \frac{\cos \ell_{12} \cos \ell_{34} - \cos \ell_{13} \cos \ell_{24}}{\cos \alpha_{12} \cos \alpha_{34} - \cos \alpha_{13} \cos \alpha_{24}} &= \frac{\cos \ell_{13} \cos \ell_{24} - \cos \ell_{14} \cos \ell_{23}}{\cos \alpha_{13} \cos \alpha_{24} - \cos \alpha_{14} \cos \alpha_{23}} \\ &= \frac{\cos \ell_{14} \cos \ell_{23} - \cos \ell_{12} \cos \ell_{34}}{\cos \alpha_{14} \cos \alpha_{23} - \cos \alpha_{12} \cos \alpha_{34}} = \frac{d}{\gamma} = \frac{\gamma'}{d'}, \end{aligned} \quad (14)$$

where $d = \det G$, $d' = \det G'$, and $\gamma = \sqrt{g_{11}g_{22}g_{33}g_{44}}$, $\gamma' = \sqrt{g'_{11}g'_{22}g'_{33}g'_{44}}$.

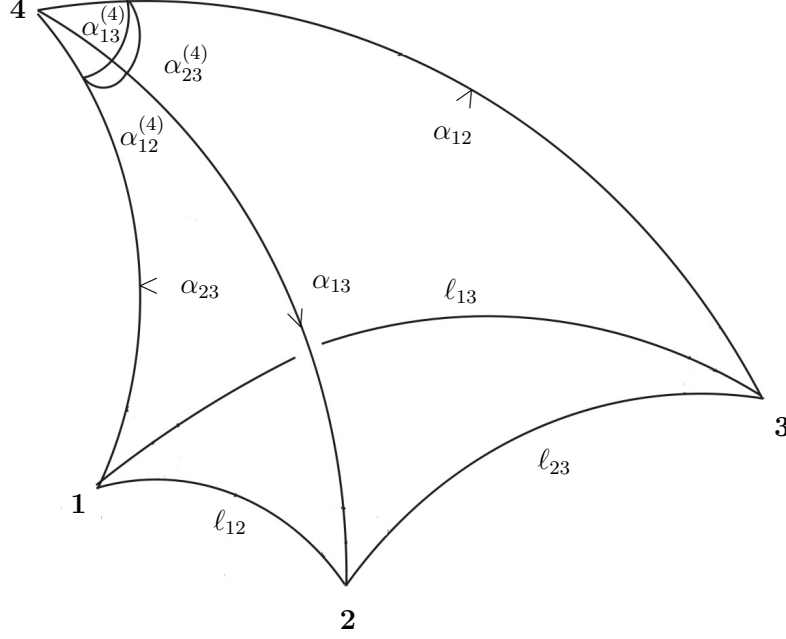


FIGURE 2. A spherical tetrahedron

3. SPHERICAL TRIANGLES AND SYMMETRIC DISCRETE DARBOUX SYSTEM

We now consider a somewhat fancy combinatorial interpretation of formulas (6) for the cosine law for spherical triangles. In this interpretation, we combinatorially assign the three inner angles of a spherical triangle and its three side lengths to the six 2-faces of an elementary cube, in such a manner that each angle and an opposite side are assigned to two opposite faces of the cube. More precisely, we assign the quantities x_{jk} to the three faces of a 3-dimensional cube parallel to the coordinate plane jk , and the quantities $y_{jk} = T_i x_{jk}$ to the three opposite faces, see Fig. 3. Here, T_i stands for the unit shift in the i -th coordinate direction.

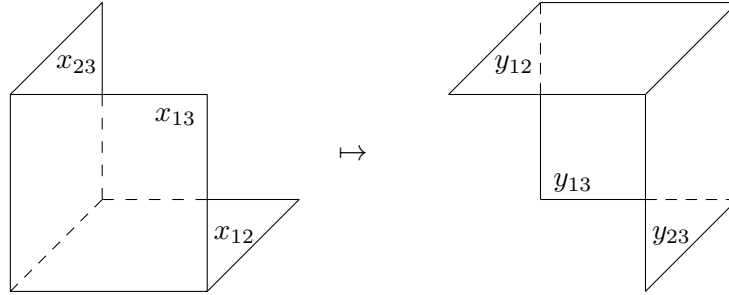


FIGURE 3. A map on an elementary 3D cube with fields assigned to 2-faces.

Thus, what we consider now is, in the terminology of [1], a 3D system with fields assigned to elementary squares, given by the formulas

$$(T_3 x_{12}, T_2 x_{13}, T_1 x_{23}) = \phi(x_{12}, x_{13}, x_{23}),$$

or, explicitly,

$$\begin{cases} T_3 x_{12} = \frac{x_{12} + x_{13}x_{23}}{\sqrt{1-x_{13}^2}\sqrt{1-x_{23}^2}}, \\ T_2 x_{13} = \frac{x_{13} + x_{12}x_{23}}{\sqrt{1-x_{12}^2}\sqrt{1-x_{23}^2}}, \\ T_1 x_{23} = \frac{x_{23} + x_{12}x_{13}}{\sqrt{1-x_{12}^2}\sqrt{1-x_{13}^2}}. \end{cases} \quad (15)$$

This 3D system will be called the *symmetric discrete Darboux system*. It can be seen as the special case (symmetric reduction) $x_{ij} = x_{ji}$ of the general *discrete Darboux system*, given by

$$T_k x_{ij} = \frac{x_{ij} + x_{ik}x_{kj}}{\sqrt{1-x_{ik}x_{ki}}\sqrt{1-x_{kj}x_{jk}}}. \quad (16)$$

System (16) is well known in the theory of discrete integrable systems of geometric origin. It seems to have appeared for the first time in [8, eq. (7.20)] as a gauge version of a simpler but less symmetric form of the discrete Darboux system,

$$T_k x_{ij} = \frac{x_{ij} + x_{ik}x_{kj}}{1 - x_{kj}x_{jk}}, \quad (17)$$

derived earlier in [3]. Either form of the discrete Darboux system describes the so called rotation coefficients of discrete conjugate nets $f : \mathbb{Z}^3 \rightarrow \mathbb{RP}^n$, i.e., nets with planar elementary quadrilaterals, see [1, p. 42]. The version of rotation coefficients satisfying system (16) has an advantage of being defined in more local terms, see [1, p. 95–96]. Besides, system (16) admits, unlike system (17), the symmetric reduction $x_{ij} = x_{ji}$, and is therefore better suited for the description of the so called symmetric conjugate nets and discrete Egorov nets, see [4]. The symmetric discrete Darboux system is a close relative of the *discrete CKP system*, see [17].

Thus, we have proved that the symmetric discrete Darboux system (15) admits an alternative interpretation in terms of spherical triangles. This interpretation leads also to a new proof of integrability of the symmetric discrete Darboux system. Let us recall that integrability of discrete 3D systems is synonymous with their 4D consistency [1]. The meaning of this notion can be described as follows.

Consider the initial value problem with the data $x_{ij} = x_{ji}$, $i, j = 1, 2, 3, 4$, prescribed at six squares adjacent to one common vertex of the 4D cube. Then the application of a 3D map like (15) to the four 3D cubes adjacent to this vertex allows us to determine all $T_k x_{ij}$. At the second stage, the map is applied to the other four 3-faces of the 4D cube, with the result being all $T_m(T_k x_{ij})$ computed in two different ways (since each of the corresponding squares is shared by two 3-faces), see Fig. 4. Now, 4D consistency of the map means that $T_m(T_k x_{ij}) = T_k(T_m x_{ij})$ for any permutation i, j, k, m of 1, 2, 3, 4 and for arbitrary initial data.

Theorem 1. *Symmetric discrete Darboux system (15) is 4D consistent.*

Proof. We will show that this is just a corollary of the geometry of spherical tetrahedra. Consider a spherical tetrahedron in \mathbb{S}^3 with the edge lengths ℓ_{ij} (connecting the vertices i and j) and dihedral angles α_{ij} (at the edges (km) , where k, m is the complement of i, j in 1, 2, 3, 4). One can take either α_{ij} or ℓ_{ij} as independent data parametrizing the spherical tetrahedron. The two sets of data are related by the cosine law for spherical tetrahedra, see section 2.2. What seems to be less known is that this cosine law for

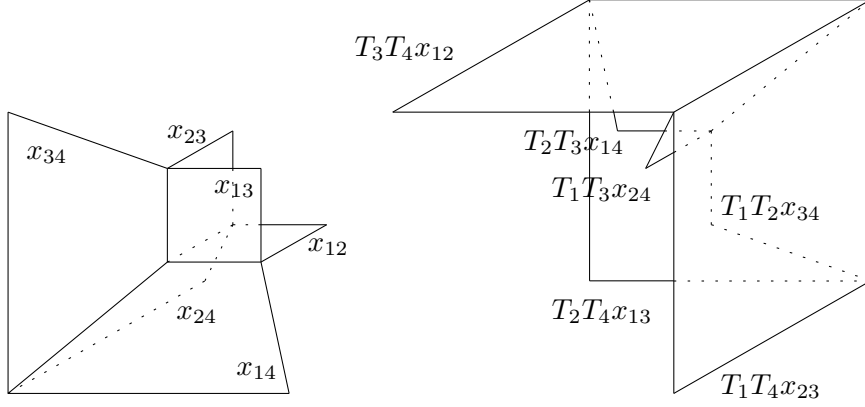


FIGURE 4. Initial data and results of two-stage application of a 4D consistent map with fields on 2-faces.

spherical tetrahedra can be derived via a two-fold application of the cosine law for the spherical triangles, if the combinatorics of this two-fold application is in agreement with Fig. 4. In other words, a spherical tetrahedron combinatorially corresponds to a 4D cube.

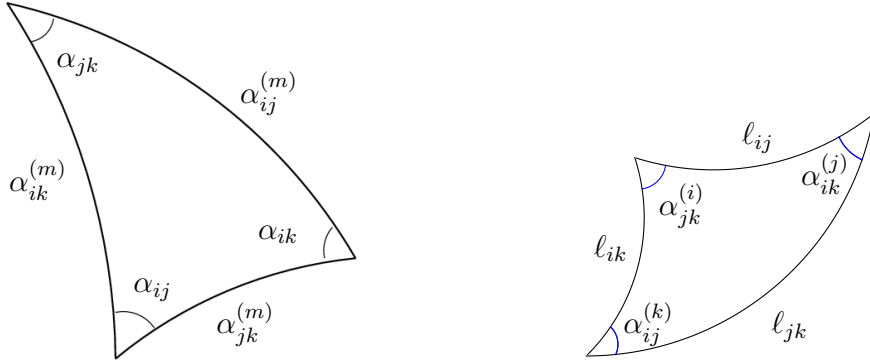


FIGURE 5. Left: Sides and angles in the spherical triangle $\text{Lk}(m)$. Right: Sides and angles in the spherical triangle $\Delta(ijk)$

To show this, we start with the dihedral angles α_{ij} as initial data. From these, we can determine all the planar angles $\alpha_{ij}^{(k)}$ (the latter is the notation for the planar angle opposite to the side (ij) in the spherical triangle $\Delta(ijk)$ with the vertices i, j, k , see Fig. 5 (right)). Indeed, draw a great 2-sphere $S_m \subset \mathbb{S}^3$ with the pole at the vertex m . It will cut the tetrahedron along a spherical triangle which is called $\text{Lk}(m)$, the link of the vertex m (see [12]). This triangle has, as one easily sees, the angles α_{ij} , α_{ik} and α_{jk} and the side lengths $\alpha_{ij}^{(m)}$, $\alpha_{ik}^{(m)}$ and $\alpha_{jk}^{(m)}$, see Fig. 5 (left). Therefore, we find:

$$(\alpha_{ij}^{(m)}, \alpha_{ik}^{(m)}, \alpha_{jk}^{(m)}) = \Phi(\alpha_{ij}, \alpha_{ik}, \alpha_{jk}).$$

In the above combinatorial interpretation, the dihedral angles α_{ij} are assigned to six 2-faces of the 4D cube, adjacent to one of its vertices. The four 3-faces (3D cubes) adjacent

to this vertex, correspond combinatorially to the four spherical triangles $Lk(m)$. The result of application of our 3D system to these four 3D cubes can be written as $\alpha_{ij}^{(m)} = T_k \alpha_{ij}$.

At the second stage, we apply the cosine law to the four faces $\Delta(i, j, k)$ of the tetrahedron, see Fig. 5 (right). In the above combinatorial interpretation, these four spherical triangles correspond to the four 3-faces of the cube adjacent to its opposite vertex. In particular, the side lengths ℓ_{ij} are assigned to the six 2-faces of the 4D cube, adjacent to this opposite vertex. We find:

$$(\ell_{ij}, \ell_{ik}, \ell_{jk}) = \Phi(\alpha_{ij}^{(k)}, \alpha_{ik}^{(j)}, \alpha_{jk}^{(i)}).$$

This can be written as $\ell_{ij} = T_k \alpha_{ij}^{(k)} = T_k(T_m \alpha_{ij})$. Clearly, each edge is shared by two 2-faces, so we get two answers for each ℓ_{ij} . Of course, they must coincide as definite geometric quantities in a given spherical tetrahedron. \square

The proof of Theorem 1 given above is completely free of computations and it is based on the existence of a spherical tetrahedron with given dihedral angles. We mention that very similar arguments were used in [11] in the opposite direction: the coincidence of the two answers for each ℓ_{ij} was demonstrated there by a direct computation and used to provide a geometric realization for a given angle structure.

Remark. In the combinatorial interpretation adopted in the present section, the map $\phi : \alpha \mapsto \ell$ is interpreted as solving a spherical triangle by its three planar angles, and combinatorially corresponds to one (directed) long diagonal of the cube. The inverse map $\phi^{-1} : \ell \mapsto \alpha$ would be interpreted as solving a spherical triangle by its three sides and would combinatorially correspond to the opposite orientation of this same long diagonal. It is easy to realize that other long diagonals of the cube would correspond to solving a spherical triangle by two sides and a planar angle enclosed by them, or by a side and two adjacent planar angles. The corresponding maps could be called *companion maps* to ϕ . It is well known that companion maps for a multidimensionally consistent map with fields on 2-faces satisfy the so called *functional tetrahedron equation*, see, e.g., [1, Sect. 6.16]. We come to a conclusion that the cosine law for spherical triangles delivers an example of solutions of the functional tetrahedron equation. According to a private communication by R. Kashaev, he was aware of this example since many years.

4. SPHERICAL TRIANGLES AND DISCRETE TIME EULER TOP

In the present section, it will be more convenient to change notation for the edges and angles of a spherical triangle to (ℓ_1, ℓ_2, ℓ_3) for $(\ell_{23}, \ell_{13}, \ell_{12})$, resp. to $(\alpha_1, \alpha_2, \alpha_3)$ for $(\alpha_{23}, \alpha_{13}, \alpha_{12})$. We study a dynamical system generated by iterations of the map ϕ from (6), resp. its inverse map ϕ^{-1} from (7), which now take the form

$$\phi : y_i = \frac{x_i + x_j x_k}{\sqrt{1 - x_j^2} \sqrt{1 - x_k^2}}, \quad \phi^{-1} : x_i = \frac{y_i - y_j y_k}{\sqrt{1 - y_j^2} \sqrt{1 - y_k^2}}. \quad (18)$$

Due to homogeneity, one can insert a small parameter ε into the map ϕ by a scaling transformation $x_i \mapsto \varepsilon x_i$, $y_i \mapsto \varepsilon y_i$ leading to

$$y_i = \frac{x_i + \varepsilon x_j x_k}{\sqrt{1 - \varepsilon^2 x_j^2} \sqrt{1 - \varepsilon^2 x_k^2}}. \quad (19)$$

Geometrically, small values of ε in (19) correspond to small values of $x_i = \cos \alpha_i$ in (18), i.e., to spherical triangles with all three angles (and all three sides) being close to $\pi/2$. The form (19) of the map ϕ makes it obvious that it can be considered as a time discretization of the famous *Euler top*, described by the following system of differential equations:

$$\dot{x}_i = x_j x_k. \quad (20)$$

The latter system can be brought to the form $\dot{x}_i = c_i x_j x_k$, involving arbitrary real parameters c_i , by a simple scaling transformation $x_i \mapsto b_i x_i$, where, however, some of b_i may be imaginary, depending on the signs of c_i .

It is well-known that system (20) is integrable in the Liouville-Arnold sense and can be solved in terms of elliptic functions, see for instance [16]. In particular, it is bi-Hamiltonian and admits integrals of motion $I_{ij} = x_i^2 - x_j^2$, two of which are functionally independent.

We will show that discretization (19) of the Euler top inherits integrability.

Theorem 2. *The map ϕ admits integrals of motion*

$$E_{ij} = \frac{1 - x_i^2}{1 - x_j^2}, \quad (21)$$

two of which are functionally independent. It has an invariant volume form given by

$$\omega = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\varphi(x)},$$

where $\varphi(x)$ is any of the functions $(1 - x_j^2)(1 - x_k^2)$ or $(1 - x_i^2)^2$ (the quotient of any two such functions is an integral of motion).

Proof. The first statement (about integrals of motion) follows immediately from the sine law (8). As for the second statement (about an invariant volume form), a direct computation leads to the following expressions:

$$\frac{\partial y_i}{\partial x_i} = \frac{1}{\sqrt{1 - x_j^2} \sqrt{1 - x_k^2}},$$

and

$$\frac{\partial y_i}{\partial x_j} = \frac{x_k + x_i x_j}{\sqrt{(1 - x_j^2)^3} \sqrt{1 - x_k^2}} = \frac{y_k}{\sqrt{1 - x_j^2} \sqrt{1 - x_k^2}} \sqrt{\frac{1 - x_i^2}{1 - x_j^2}}.$$

The second factor in the last formula can be interpreted as a conjugation of the Jacobi matrix $\partial y / \partial x$ by the diagonal matrix with entries $\sqrt{1 - x_i^2}$, and is therefore irrelevant for the computation of its determinant, which is thus equal to

$$\det \left(\frac{\partial y}{\partial x} \right) = \frac{1}{(1 - x_1^2)(1 - x_2^2)(1 - x_3^2)} \begin{vmatrix} 1 & y_3 & y_2 \\ y_3 & 1 & y_1 \\ y_2 & y_1 & 1 \end{vmatrix} = \frac{d'}{(1 - x_1^2)(1 - x_2^2)(1 - x_3^2)}.$$

According to the sine law (8), the latter expression is equal to

$$\det \left(\frac{\partial y}{\partial x} \right) = \frac{(1 - y_j^2)(1 - y_k^2)}{(1 - x_j^2)(1 - x_k^2)},$$

which finishes the proof. \square

Corollary 1. *The map ϕ is Poisson with respect to Poisson brackets (bi-vector fields) obtained by contraction of the tri-vector field $\varphi(x)\partial_1 \wedge \partial_2 \wedge \partial_3$ with any of the one-forms $dE_{ij}(x)$. Explicitly, a family of compatible invariant Poisson brackets is given by*

$$\{x_i, x_j\} = C_i x_k (1 - x_j^2) - C_j x_k (1 - x_i^2),$$

where C_1, C_2, C_3 are arbitrary constants. Thus, the map ϕ is bi-Hamiltonian and is completely integrable in the Liouville-Arnold sense.

A somewhat unpleasant feature of the map ϕ is its non-rational character, due to the square roots in the denominators. A striking and unexpected fact is that the second iterate of ϕ is a rational (actually, a birational) map, which appeared previously in the literature on a different occasion.

Theorem 3. *On the set $\tau \cap \phi^{-1}(\tau)$, where the second iterate of ϕ is defined, this second iterate $\phi^2 : x \mapsto \tilde{x}$ is a birational map given by*

$$\tilde{x}_i = \frac{x_i + 2x_j x_k + x_i(-x_i^2 + x_j^2 + x_k^2)}{1 - (x_1^2 + x_2^2 + x_3^2) - 2x_1 x_2 x_3}. \quad (22)$$

These functions are the (unique) solutions of the following system of equations:

$$\tilde{x}_i - x_i = \tilde{x}_j x_k + x_j \tilde{x}_k, \quad (23)$$

which constitute the Hirota-Kimura discretization of the Euler top [5].

Proof. We let $\phi(x) = y$, $\phi(y) = \tilde{x}$. According to formulas (18), we have

$$x_i = \frac{y_i - y_j y_k}{\sqrt{1 - y_j^2} \sqrt{1 - y_k^2}}, \quad \tilde{x}_i = \frac{y_i + y_j y_k}{\sqrt{1 - y_j^2} \sqrt{1 - y_k^2}}. \quad (24)$$

Using (24), we find, on one hand,

$$\tilde{x}_i - x_i = \frac{2y_j y_k}{\sqrt{1 - y_j^2} \sqrt{1 - y_k^2}},$$

and, on the other hand,

$$\tilde{x}_j x_k + x_j \tilde{x}_k = \frac{(y_j + y_k y_i)(y_k - y_i y_j) + (y_j - y_k y_i)(y_k + y_i y_j)}{(1 - y_i^2) \sqrt{1 - y_j^2} \sqrt{1 - y_k^2}} = \frac{2y_j y_k}{\sqrt{1 - y_j^2} \sqrt{1 - y_k^2}}.$$

This proves equations (23) for ϕ^2 . These equations are linear with respect to $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$, and thus they can be solved to give the explicit birational map (22). \square

Actually, the integrability attributes of the map ϕ given in Theorem 2 and Corollary 1 were found previously for the rational map ϕ^2 : integrals in [5] and the invariant volume form along with the bi-Hamiltonian structure in [15]. We refer also to [6, 13, 14] for our recent study on Hirota-Kimura-type discretizations.

While the map Φ relates the data within one spherical triangle, a natural interpretation of the map Φ^2 is given by the following map between spherical triangles.

Definition 1. *Let \triangle be a spherical triangle with the angles α_i and the sides ℓ_i . The switched triangle $\tilde{\triangle} = S(\triangle)$ is the spherical triangle with inner angles*

$$\tilde{\alpha}_i = \ell_i \quad (25)$$

(whenever such a triangle exists). The switched triangle is defined up to a transformation from $O(3)$.

Clearly, $S(\Delta)$ exists if and only if $\ell \in \mathcal{T} \cap \Phi(\mathcal{T})$, i.e., if and only if $\alpha \in \Phi^{-1}(\mathcal{T}) \cap \mathcal{T}$. In this case, we have $\tilde{\ell} = \Phi^2(\alpha)$. Thus, the quantities $\tilde{x}_i = \cos \tilde{\ell}_i$ are related to $x_i = \cos \alpha_i$ by the Hirota-Kimura discrete time Euler top equations (22) or (23). By abusing notations slightly, we write $(\tilde{\alpha}, \tilde{\ell}) = S(\alpha, \ell)$. By induction, we have:

$$S^n(\alpha, \ell) = (\Phi^n(\alpha), \Phi^{n+1}(\alpha)).$$

Therefore, $S^n(\Delta)$ is defined iff $\alpha \in \mathcal{T} \cap \Phi^{-1}(\mathcal{T}) \cap \dots \cap \Phi^{-n}(\mathcal{T})$. For any n , the latter domain contains an open neighborhood of the point $(\pi/2, \pi/2, \pi/2)$.

When talking about the switched triangles to our colleagues, we have been often asked: “But isn’t this just the polar triangle?” The answer is: “Of course not, since the switch generates a non-trivial dynamical system, while the polarity is an involution”. Nevertheless, it turns out that these two transformations of spherical triangles are closely related. We denote the angles and the sides of the polar triangle $\Delta^* = P(\Delta)$ by $(\alpha^*, \ell^*) = P(\alpha, \ell)$. They are given by

$$\alpha_i^* = \pi - \ell_i, \quad \ell_i^* = \pi - \alpha_i. \quad (26)$$

Clearly, in the latter interpretation, P maps $\mathcal{T} \times \mathcal{T}^*$ to itself. Comparing formulas (25) and (26), we are led to the following definition.

Definition 2. Let Δ be a spherical triangle with the angles α_i and the sides ℓ_i . The side flip $\hat{\Delta} = F(\Delta)$ is the spherical triangle with sides

$$\hat{\ell}_i = \pi - \ell_i \quad (27)$$

(whenever such a triangle exists). The side flip $F(\Delta)$ is defined up to a transformation from $O(3)$. The map F is an involution (when considered on equivalence classes modulo $O(3)$).

An immediate consequence of the above definition is the following claim.

Theorem 4. The switch of spherical triangles, considered as a map on equivalence classes modulo $O(3)$, can be represented as a composition of two involutions, namely as the side flip followed by the polarity transformation: $S = P \circ F$.

The angles $\hat{\alpha}_i$ of the side flipped triangle $\hat{\Delta} = F(\Delta)$ are given by $\hat{\alpha} = \pi - \Phi^2(\alpha)$. The quantities $x_i = \cos \alpha_i$ and $\hat{x}_i = \cos \hat{\alpha}_i$ satisfy

$$\hat{x}_i + x_i = \hat{x}_j x_k + x_j \hat{x}_k, \quad (28)$$

or, explicitly,

$$\hat{x}_i = -\frac{x_i + 2x_j x_k + x_i(-x_i^2 + x_j^2 + x_k^2)}{1 - (x_1^2 + x_2^2 + x_3^2) - 2x_1 x_2 x_3}. \quad (29)$$

The map $f : x \mapsto \hat{x}$ defined by (29) is a birational involution.

It is very easy to find integrals of this map. Indeed, from the defining relation (27) yielding $\cos \hat{\ell}_i = -\cos \ell_i$ and the cosine law (3) there follows:

$$\frac{\cos \hat{\alpha}_i + \cos \hat{\alpha}_j \cos \hat{\alpha}_k}{\sin \hat{\alpha}_j \sin \hat{\alpha}_k} = -\frac{\cos \alpha_i + \cos \alpha_j \cos \alpha_k}{\sin \alpha_j \sin \alpha_k}.$$

As a consequence, one gets three functionally independent conserved quantities of the map (29):

$$\cos^2 \ell_i = \frac{(x_i + x_j x_k)^2}{(1 - x_j^2)(1 - x_k^2)} \quad \Leftrightarrow \quad \sin^2 \ell_i = \frac{d}{(1 - x_j^2)(1 - x_k^2)},$$

with the quantity d defined by (9). Note that the quotients $E_{ij} = \sin^2 \ell_i / \sin^2 \ell_j$ depend only on x_i^2 rather than on x_i , and are therefore integrals of motion for the map $\phi^2 = i \circ f$ (these are nothing but functions (21)).

Map (29) appeared in an old paper [7] by H. Jonas. Actually, he considered a closely related *angle flip* $\triangle \mapsto \bar{\triangle}$ of spherical triangles, defined by the relation $\bar{\alpha}_i = \pi - \alpha_i$, and proved that the cosines of the side lengths of the spherical triangles $\triangle, \bar{\triangle}$ are related by

$$\bar{x}_i + x_i + \bar{x}_j x_k + x_j \bar{x}_k = 0. \quad (30)$$

The latter map is conjugated to (28) via $i : x \mapsto -x$. Jonas' results include integrals of the map (30) and its solution in terms of elliptic functions.

5. SPHERICAL TETRAHEDRA AND DISCRETIZATION OF TWO LINEARLY COUPLED EULER TOPS

We now consider the algebraic form of the cosine law for spherical tetrahedra as a dynamical system. Recall that it is given by

$$y_{ij} = \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}}, \quad (31)$$

with the expressions for the polynomials g_{ij} given in (11–12). This map, which we denote by ψ , can be considered as a time discretization of the following system of ordinary differential equations:

$$\dot{x}_{ij} = x_{ik}x_{jk} + x_{im}x_{jm}. \quad (32)$$

Here, recall, (i, j, k, m) stands for an arbitrary permutation of $(1, 2, 3, 4)$. This six-dimensional system turns out to consist of two linearly coupled copies of the Euler top. Namely, each group of three variables $p_{ij} = x_{ij} + x_{km}$ and $q_{ij} = x_{ij} - x_{km}$ with $ij = 12, 13, 23$, satisfies its own copy of the Euler top equations:

$$\dot{p}_{ij} = p_{ik}p_{jk}, \quad \dot{q}_{ij} = q_{ik}q_{jk}. \quad (33)$$

As a consequence, equations (32) are integrable in the Liouville-Arnold sense, with four independent integrals of motion which can be chosen as

$$\begin{aligned} p_{12}^2 - p_{13}^2 &= (x_{12} + x_{34})^2 - (x_{13} + x_{24})^2, & p_{13}^2 - p_{23}^2 &= (x_{13} + x_{24})^2 - (x_{23} + x_{14})^2, \\ q_{12}^2 - q_{13}^2 &= (x_{12} - x_{34})^2 - (x_{13} - x_{24})^2, & q_{13}^2 - q_{23}^2 &= (x_{13} - x_{24})^2 - (x_{23} - x_{14})^2, \end{aligned}$$

or, alternatively, as

$$\begin{aligned} x_{12}^2 + x_{34}^2 - x_{13}^2 - x_{24}^2, & \quad x_{13}^2 + x_{24}^2 - x_{23}^2 - x_{14}^2, \\ x_{12}x_{34} - x_{13}x_{24}, & \quad x_{13}x_{24} - x_{23}x_{14}. \end{aligned}$$

Theorem 5. *The map ψ admits four independent integrals of motion*

$$\begin{aligned} \frac{(1 - x_{13}^2)(1 - x_{24}^2)}{(1 - x_{12}^2)(1 - x_{34}^2)}, & \quad \frac{(1 - x_{23}^2)(1 - x_{14}^2)}{(1 - x_{12}^2)(1 - x_{34}^2)}, \\ \frac{x_{12}x_{34} - x_{13}x_{24}}{\sqrt{(1 - x_{12}^2)(1 - x_{34}^2)}}, & \quad \frac{x_{13}x_{24} - x_{23}x_{14}}{\sqrt{(1 - x_{12}^2)(1 - x_{34}^2)}}. \end{aligned}$$

It has an invariant volume form given by

$$\omega = \frac{dx_{12} \wedge dx_{13} \wedge dx_{23} \wedge dx_{14} \wedge dx_{24} \wedge dx_{34}}{\varphi(x)},$$

where $\varphi(x)$ is any of the functions $(1 - x_{ij}^2)^{5/2}(1 - x_{km}^2)^{5/2}$.

Proof. The claim about integrals of motion follows directly from the sine laws (13–14). Let us turn to the proof of the claim about the invariant volume form. Our aim is to compute the determinant of the Jacobi matrix $(\partial y_{ij}/\partial x_{i'j'})$. We use the following ordering of the index pairs: 12, 13, 23, 14, 24, 34. The entries of the Jacobi matrix are given by the following expressions:

$$\frac{\partial y_{ij}}{\partial x_{ij}} = \frac{1 - x_{km}^2}{\sqrt{g_{ii}g_{jj}}}, \quad (34)$$

$$\frac{\partial y_{ij}}{\partial x_{ik}} = \frac{1 - x_{km}^2}{\sqrt{g_{ii}g_{jj}}} y_{jk} \sqrt{\frac{g_{kk}}{g_{jj}}}, \quad (35)$$

$$\frac{\partial y_{ij}}{\partial x_{km}} = \frac{1 - x_{km}^2}{\sqrt{g_{ii}g_{jj}}} p_{km}^{ij} \sqrt{\frac{g_{kk}g_{mm}}{g_{ii}g_{jj}}}, \quad (36)$$

where

$$p_{km}^{ij} = \frac{y_{ik}y_{jm} + y_{im}y_{jk} - y_{ij}y_{ik}y_{im} - y_{ij}y_{jk}y_{jm}}{1 - y_{ij}^2}.$$

Indeed, the first two expressions, (34–35), are easily obtained by a direct computation, while the third one, (36), is a result of [12]. The last factors in the above expressions can be interpreted as a conjugation of the Jacobi matrix $(\partial y_{ij}/\partial x_{i'j'})$ by a diagonal matrix

$$\text{diag} \left(1, \sqrt{\frac{g_{33}}{g_{22}}}, \sqrt{\frac{g_{33}}{g_{11}}}, \sqrt{\frac{g_{44}}{g_{22}}}, \sqrt{\frac{g_{44}}{g_{11}}}, \sqrt{\frac{g_{33}g_{44}}{g_{11}g_{22}}} \right),$$

and are therefore inessential for the computation of the determinant. We come to the following result:

$$\det \left(\frac{\partial y}{\partial x} \right) = \frac{\prod_{k < m} (1 - x_{km}^2)}{(g_{11}g_{22}g_{33}g_{44})^{3/2}} \begin{vmatrix} 1 & y_{23} & y_{13} & y_{24} & y_{14} & p_{12}^{34} \\ y_{23} & 1 & y_{12} & y_{34} & p_{13}^{24} & y_{14} \\ y_{13} & y_{12} & 1 & p_{23}^{14} & y_{34} & y_{24} \\ y_{24} & y_{34} & p_{14}^{23} & 1 & y_{12} & y_{13} \\ y_{14} & p_{24}^{13} & y_{34} & y_{12} & 1 & y_{23} \\ p_{34}^{12} & y_{14} & y_{24} & y_{13} & y_{23} & 1 \end{vmatrix}.$$

A direct computation with a symbolic manipulation system like Maple leads to the following factorized expression for the determinant on the right-hand side:

$$\begin{vmatrix} 1 & y_{23} & y_{13} & y_{24} & y_{14} & p_{12}^{34} \\ y_{23} & 1 & y_{12} & y_{34} & p_{13}^{24} & y_{14} \\ y_{13} & y_{12} & 1 & p_{23}^{14} & y_{34} & y_{24} \\ y_{24} & y_{34} & p_{14}^{23} & 1 & y_{12} & y_{13} \\ y_{14} & p_{24}^{13} & y_{34} & y_{12} & 1 & y_{23} \\ p_{34}^{12} & y_{14} & y_{24} & y_{13} & y_{23} & 1 \end{vmatrix} = \frac{g'_{11}g'_{22}g'_{33}g'_{44}}{\prod_{i < j} (1 - y_{ij}^2)} d'.$$

Therefore,

$$\det \left(\frac{\partial y}{\partial x} \right) = \frac{\prod_{k < m} (1 - x_{km}^2)}{\prod_{i < j} (1 - y_{ij}^2)} \frac{(g'_{11}g'_{22}g'_{33}g'_{44})^{3/2}}{(g_{11}g_{22}g_{33}g_{44})^{3/2}} \frac{d'}{\gamma'}.$$

At the next step, we use the following result (proved below):

$$\frac{(g'_{11}g'_{22}g'_{33}g'_{44})^{3/2}}{(g_{11}g_{22}g_{33}g_{44})^{3/2}} = \frac{\prod_{i < j} (1 - y_{ij}^2)^2}{\prod_{k < m} (1 - x_{km}^2)^2}. \quad (37)$$

As a consequence, we find with the help of (13):

$$\det \left(\frac{\partial y}{\partial x} \right) = \frac{\prod_{i < j} (1 - y_{ij}^2)}{\prod_{k < m} (1 - x_{km}^2)} \frac{d'}{\gamma'} = \left(\frac{\gamma'}{d'} \right)^6 \frac{d'}{\gamma'} = \left(\frac{\gamma'}{d'} \right)^5,$$

and one more reference to relations (13) proves the claim about the invariant volume form.

To finish the proof we need to prove formula (37). Comparing expression (9) with (12), we see that g_{ii} is the analogue of the quantity d for the spherical triangle with the angles α_{jk} , α_{jm} , α_{km} , i.e., for the triangle $\text{Lk}(i)$. Then, from (10), we obtain:

$$g_{ii}^{1/2} = \sin \alpha_{jk}^{(i)} \sin \alpha_{jm} \sin \alpha_{km}.$$

Multiplying all three such expressions for $\text{Lk}(i)$, we find:

$$g_{ii}^{3/2} = \sin \alpha_{jk}^{(i)} \sin \alpha_{jm}^{(i)} \sin \alpha_{km}^{(i)} \sin^2 \alpha_{jk} \sin^2 \alpha_{jm} \sin^2 \alpha_{km}.$$

Similarly, g'_{ii} is the analogue of the quantity d' for the spherical triangle with the sides ℓ_{jk} , ℓ_{jm} , ℓ_{km} , i.e., for the triangle $\Delta(j, k, m)$. From (10), we obtain:

$$(g'_{ii})^{1/2} = \sin \alpha_{jk}^{(m)} \sin \ell_{jm} \sin \ell_{km}.$$

Multiplying all three such expressions for $\Delta(j, k, m)$, we find:

$$(g'_{ii})^{3/2} = \sin \alpha_{jk}^{(m)} \sin \alpha_{jm}^{(k)} \sin \alpha_{km}^{(j)} \sin^2 \ell_{jk} \sin^2 \ell_{jm} \sin^2 \ell_{km}.$$

As a consequence,

$$\begin{aligned} (g_{11}g_{22}g_{33}g_{44})^{3/2} &= \left(\prod_{i=1}^4 \prod_{j < k} \sin \alpha_{jk}^{(i)} \right) \prod_{j < k} \sin^4 \alpha_{jk}, \\ (g'_{11}g'_{22}g'_{33}g'_{44})^{3/2} &= \left(\prod_{m=1}^4 \prod_{j < k} \sin \alpha_{jk}^{(m)} \right) \prod_{j < k} \sin^4 \ell_{jk}. \end{aligned}$$

Indeed, each dihedral angle α_{jk} participates as a planar angle of two triangles $\text{Lk}(i)$ and $\text{Lk}(m)$, and each side ℓ_{jk} is shared by two triangles $\Delta(j, k, m)$ and $\Delta(i, j, k)$. This proves (37). \square

Remark. According to the famous Schläfli formula (see, e.g., [12, 18]), if $V = V(\alpha)$ is the volume of the spherical tetrahedron with the dihedral angles α_{ij} , then $dV = \frac{1}{2} \sum_{ij} \ell_{km} d\alpha_{ij}$. Thus, our proof of Theorem 5 actually leads to an evaluation of the determinant of the Hesse matrix of the volume function, $\det(\partial^2 V / \partial \alpha_{ij} \partial \alpha_{i'j'})$. We are not aware of such an evaluation in the literature.

Remark. The reduction of the system of ordinary differential equations (32) to two copies of the Euler top (33) yields that the original system admits two invariant 3-forms, $dp_{12} \wedge dp_{13} \wedge dp_{23}$ and $dq_{12} \wedge dq_{13} \wedge dq_{23}$. We conjecture that the discretization ψ also admits an invariant 3-form. This property is stronger than existence of an invariant volume form established in Theorem 5. If true, this 3-form (or, better, the corresponding 3-vector field)

would yield, by contraction with gradients of integrals of motion, a family of compatible Poisson structures for ψ .

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